EXTREMALS FOR LOGARITHMIC HARDY-LITTLEWOOD-SOBOLEV INEQUALITIES ON COMPACT MANIFOLDS

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Abstract. Let M be a closed, connected surface and let Γ be a conformal class of metrics on M with each metric normalized to have area V. For a metric $g \in \Gamma$, denote the area element by dV and the Laplace-Beltrami operator by Δ_g. We define the Robin mass $m(x)$ at the point $x \in M$ to be the value of the Green's function $G(x,y)$ at $y = x$ after the logarithmic singularity has been subtracted off. The regularized trace of Δ_g^{-1} is then defined by $\operatorname{trace}\Delta^{-1} = \int_M m\,dV$. (This essentially agrees with the zeta functional regularization and is thus a spectral invariant.) Let $\Delta_{S^2,V}$ be the Laplace-Beltrami operator on the round sphere of volume V. We show that if there exists $g \in \Gamma$ with $\operatorname{trace}\Delta_g^{-1} < \operatorname{trace}\Delta_{S^2,V}^{-1}$ then the minimum of $\operatorname{trace}\Delta^{-1}$ over Γ is attained by a metric in Γ for which the Robin mass is constant. Otherwise, the minimum of $\operatorname{trace}\Delta^{-1}$ over Γ is equal to $\operatorname{trace}\Delta_{S^2,V}^{-1}$. In fact we prove these results in the general setting where M is an n dimensional closed, connected manifold and the Laplace-Beltrami operator is replaced by any non-negative elliptic operator A of degree n which is conformally covariant in the sense that for the metric g we have $A_{F^{2/n}g} = F^{-1}A_g$. In this case the role of $\Delta_{S^2,V}$ is assumed by the *Paneitz* or *GJMS operator* on the round n-sphere of volume V. Explicitly these results are logarithmic HLS inequalities for (M,g). By duality we obtain analogs of the Onofri-Beckner theorem.

Section 1. Introduction and Results.

Let S^n denote the standard unit sphere in $\mathbb{R}^{n+1}$ with normalized volume element $d\sigma$. In [CL] and [Be] the following endpoint Hardy-Littlewood-Sobolev inequality was established:

Sharp logarithmic Hardy-Littlewood-Sobolev inequality on the sphere. *The inequality*

$$\frac{2}{n!}\int_{S^n} F\log F\,d\sigma \;-\; \int_{S^n} F\Box^{-1}F\,d\sigma \;\geq\; 0 \tag{1.1}$$

holds for all functions $F : S^n \to [0,\infty)$ with $\int_{S^n} F\,d\sigma = 1$, such that $\int_{S^n} F\log F\,d\sigma$ is finite. Moreover equality is attained exactly when F is the Jacobian of a conformal transformation of S^n.

Here, the operator $\Box$ is a natural operator of order n on S^n given by its action on a spherical harmonic Y_k of degree k on S^n by

$$\Box Y_k \;=\; k(k+1)\dots(k+n-1)Y_k.$$

When $n = 2$, $\Box$ is just the Laplace-Beltrami operator, when $n = 4$ it is the Paneitz operator, and for general even n it is the GJMS operator. The inverse

$$\Box^{-1}Y_k \;=\; \begin{cases} Y_k/(k(k+1)\dots(k+n-1)), & k > 0 \\ 0 & k = 0 \end{cases}$$

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can be written as an integral operator. Indeed, writing $|x|$ for the Euclidean norm of $x \in \mathbb{R}^{n+1}$, we have

$$\Box^{-1}F(x) \;=\; -\frac{2}{(n-1)!}\int_{S^n} \log|x-y|\;(F(y)-\sigma_F)\,d\sigma(y), \qquad\qquad \text{where } \sigma_F := \int_{S^n} F\,d\sigma.$$

The inequality (1.1) is dual to the following inequality, see [Be], [CL], CY], which in 2 dimensions is known as Onofri's inequality [On].

Beckner's inequality. *If u is in the Sobolev space $L^2_{n/2}(S^n)$ (functions on S^n with $n/2$ derivatives in L^2), then*

$$\frac{1}{2n!}\int_{S^n} u\Box u\,d\sigma \;+\; \int_{S^n} u\,d\sigma \;-\; \log\left(\int_{S^n} e^u\,d\sigma\right) \;\geq\; 0. \tag{1.2}$$

Moreover equality is attained exactly when e^u is a multiple of the Jacobian of a conformal transformation of S^n.

For some related inequalities, see [Ad], [Au], [CC], [F], [L], [Mor2], [Mos], [Ok*].

Now the inequalities (1.1) and (1.2) have geometric interpretations. The best known is the case of Onofri's inequality, for which the interpretation involves the zeta-regularized determinant of the Laplace-Beltrami operator.

Polyakov-Ray-Singer formula. *Let M be a closed surface with metric g, area element dV, and let K be the Gaussian curvature of M.*

$$\log\det\Delta_{e^u g} \;-\; \log\det\Delta_g \;=\; -\frac{1}{12\pi}\left(\frac{1}{4}\int_M |\nabla u|^2\,dV \;+\; \int_M Ku\,dV\right) \;+\; \log\int_M e^u\,\frac{dV}{V}. \tag{1.3}$$

We see that when g is the standard metric on S^2 and the area of $(M, e^u g)$ equals that of (M, g), then the Polyakov functional in (1.3) is a multiple of the left hand side of (1.2). Hence Onofri's inequality can be interpreted as saying that among all metrics conformal to the standard metric on S^2 having the same area, the standard metric attains the maximum value of $\det\Delta$. See [OPS1]. The analog of the Polyakov-Ray-Singer formula for the determinant of $\Box$ in 4 dimensions was computed by Branson [Br]. On S^4, the Beckner functional does occur in this formula, plus another functional which is also minimized at the standard metric. For other extremal results concerning determinants, see for example [BrØ], [CQ], [CY], [HZ], [Ok1], [Ok2], [OW], [OPS2], [R].

The inequality (1.1) on the other hand is related to the the regularized trace of $\Box^{-1}$. Indeed, suppose g is the standard metric on S^n and F is a positive smooth function on S^n. Then for the metric $F^{2/n}g$, define the operator $\Box$ by $\Box_{F^{2/n}g} = F^{-1}\Box_g$. The leading order term of $\Box$ agrees with $\Delta^{n/2}$ where Δ is the Laplace-Beltrami operator. When n is even, the differential operator $\Box$ is known as the GJMS operator and is given locally by a geometrically invariant formula. The following formula was proved in [Mor1] for the zeta regularization, and then [Mor2], [St1], [St2]. (See also (1.5) and (1.6).)

Conformal change of $\operatorname{trace}\Box^{-1}$ **on the sphere.** *If g is the standard metric on S^n and $\int_M F\,d\sigma = 1$, then*

$$\operatorname{trace}\Box^{-1}_{F^{2/n}g} \;-\; \operatorname{trace}\Box^{-1}_g \;=\; \frac{2}{n!}\int_{S^n} F\log F\,d\sigma \;-\; \int_{S^n} F\Box^{-1}F\,d\sigma.$$

Morpurgo [Mor1] then interpreted the sharp logarithmic HLS inequality as saying that among all metrics conformal to g with the same volume, the standard metric minimizes trace $\Box^{-1}$. We also note that Steiner [St1], [St2] gave an interpretation of the sharp logarithmic HLS inequality as an analog of the Riemannian positive mass theorem, see [SY], [AH].

In this paper, we seek to understand analogs of (1.1), (1.2) on general closed manifolds. Let M be a smooth compact n dimensional manifold without boundary with a Riemannian metric g_0. We denote the volume element of g_0 by dV_0 and the volume of (M, g_0) by V. Let Γ be the space of metrics conformal to g_0 which have volume V. Suppose that the operator A_{g_0} is *of type* $\Delta^{n/2}$, meaning that it is a classical elliptic pseudodifferential operator on M of degree n, which is non-negative, self-adjoint with respect to dV_0, and has null space precisely the constants, and moreover its leading order term agrees with that of $\Delta^{n/2}$ where Δ is the Laplace-Beltrami operator. For $g = F^{2/n} g_0$ in Γ, we set

$$A_g \;=\; F^{-1} A_{g_0}.$$

Then trivially, for any two metrics g and $\tilde{g} = \tilde{F}^{2/n} g$ in Γ, we have

$$A_{\tilde{g}} \;=\; \tilde{F}^{-1} A_g.$$

For example, we can take A to be the GJMS operator $\Box$ provided it is non-negative and has null space equal to the constants, see [C], [FG], [GJMS], [GZ], [Gu]. However, we can work with any operators A_g constructed as above. Now fix $g \in \Gamma$ and let dV denote the volume element for the metric g. The operator $A = A_g$ is self-adjoint with respect to dV. Define A^{-1} to be the linear operator equal to the inverse of A on the orthogonal complement of the locally constant functions, and to equal zero on the locally constant functions. The Green's function for A is the function on $M \times M$ which satisfies

$$A^{-1}\phi \;=\; \int_M G(p,q)\phi(q)\, dV(q).$$

Writing $d_g(p,q)$ for the Riemannian distance from p to q in the metric g, the function $G(p,q)$ has an expansion at the diagonal of the form

$$G(p,q) = -\frac{2n}{\gamma_n} \log d_g(p,q) \;+\; m(p) \;+\; O(d_g(p,q)),$$

where

$$\gamma_n \;=\; n!\,\omega_n, \qquad\qquad \omega_n = \text{ volume of standard } n \text{ sphere }.$$

The quantity $m(p) = m_g(p)$ is called the *Robin mass* at the point $p \in M$. There are two natural ways to regularize the trace of A^{-1}. One is to use the spectrum to take the *zeta regularization*. It is simpler, however, to define

$$\operatorname{trace} A^{-1} \;=\; \int_M m\, dV.$$

For our results, the choice of definition is irrelevant since the two definitions differ by $c_n V$ where c_n is a universal constant depending only on dimension, see [Mor2], [St1], [St2], and also the appendix of this paper. We note that recently Doyle and Steiner [DS1] gave a probabilistic interpretation of $m(x)$ and trace Δ^{-1} on closed surfaces.

Theorem 1. *Writing $\Box_{S^n,V}$ for the GJMS operator on the round n-sphere of volume V, we have*

$$\inf_{g\in\Gamma} \operatorname{trace} A_g^{-1} \ \leq \ \operatorname{trace} \Box_{S^n,V}^{-1}. \tag{1.4}$$

Moreover, if M is connected and the inequality (1.4) is strict, then the infimum on the left hand side is attained on Γ at a metric for which $m(p)$ is constant.

Remarks. 1. Theorem 1 has some similarities with the Yamabe theorem, where the mass is replaced by the scalar curvature, see for example [Sc], [Y]. In particular, we have a lack of compactness in this problem which we get around using a method similar to [Y].
2. If the operator A had trivial null space, then we would have $m_{F^{2/n}g} = m_g + \frac{2}{\gamma_n}\log F$, and the existence of constant mass metrics would be obvious. This can be compared with [H] where the existence of constant mass metrics for the conformal Laplacian is immediate. The fact that A has non-trivial null space introduces a logarithmic HLS inequality into our analysis of constant mass metrics.
3. If the closed manifold M has more than one component, and if a minimizer for trace Δ^{-1} can be found for each individual component when the volume of that component is fixed and the metric varied conformally, then a minimizer can be found on M by scaling the minimizers for each component appropriately. The appropriate scaling can be found easily using a Lagrange multiplier.
4. In [Ok3], it is shown that the inequality in (1.4) is strict for any 2-torus.

From now on fix a Riemannian metric g on M with volume element dV, and for a positive function F on M define

$$V_F \ = \ \int_M F\,dV.$$

We introduce the space

$$(L\log L)^+(M) \ = \ \left\{F : M \to [0,\infty) \ : \ F \text{ is measurable}, \ \int_M F\log F\,dV < \infty\right\}.$$

Conformal change of trace A^{-1}**.** *If F is smooth and positive on M then*

$$\operatorname{trace} A_{F^{2/n}g}^{-1} \ = \ \mu(M,g,F), \tag{1.5}$$

where $\mu(M,g,F)$ is defined for $F\in(L\log L)^+(M)$ by

$$\mu(M,g,F) \ = \ \int_M mF\,dV \ + \ \frac{2}{\gamma_n}\int_M F\log F\,dV \ - \ \frac{1}{V_F}\int_M FA^{-1}F\,dV, \tag{1.6}$$

where m is the mass for A_g.

See [Mor1], [Mor2], [St1], [St2]. We give slightly different proof of (1.5) in Section 3. We can rewrite Theorem 1 in the following form.

Theorem 1′. *Let M be a smooth n-dimensional manifold M with Riemannian metric g. Let A be an operator on M of type $\Delta^{n/2}$ with Robin mass m. Then*

$$\inf_{\substack{V_F=V\\ F\in(L\log L)^+(M)}} \mu(M,g,F) \leq \ \operatorname{trace} \Box_{S^n,V}^{-1}. \tag{1.7}$$

Moreover, if M is connected and the inequality in (1.7) is strict, the infimum on the left hand side is attained by some smooth positive $F \in (L\log L)^+(M)$ with $V_F = V$ which satisfies the pseudodifferential equation (or partial differential equation if A is a partial differential operator)

$$A(\log F)(p) \;=\; \gamma_n \left(\frac{F(p)}{V} \;-\; \frac{Am(p)}{2} \;-\; 1 \right). \tag{1.8}$$

Furthermore, if $F = 1$ attains the infimum on the left hand side of (1.7), then for all $F \in (L\log L)^+(M)$ with $V_F = V$,

$$\frac{2}{\gamma_n} \int_M F \log F \, dV \;-\; \frac{1}{V} \int_M F A^{-1} F \, dV \;\geq\; 0. \tag{1.9}$$

Remarks. 1. If F is a local minimizer for $\mu(M,g,F)$ among functions $F \in (L\log L)^+(M)$ with $V_F = V$, then F is smooth and positive and satisfies (1.8). Just follow (2.20)-(2.22) with $\varepsilon = 0$.
2. The functional $\mu(M,g,F)$ changes when either g or F is scaled. We have stated (1.7) when $V_F = V$, but in view of (1.5) this is unnecessary. Indeed, for any positive constant V' we have

$$\inf_{\substack{V_F = V' \\ F \in (L\log L)^+(M)}} \mu(M,g,F) \leq \; \mathrm{trace}\, \Box^{-1}_{S^n, V'}.$$

However, we do need to assume $V_F = V$ in (1.9).

Definition. For $p \in M$, write $B_g(p,\delta) = \{q \in M : d_g(p,q) < \delta\}$.

The inequality (1.7) follows from the next more precise result.

Theorem 2. *Let M be a smooth n-dimensional manifold M with Riemannian metric g. Let A be an operator on M of type $\Delta^{n/2}$ with Robin function m. Then for each point $p \in M$,*

$$\lim_{\delta \to 0} \inf_{\substack{V_F = V \\ F \in (L\log L)^+(M) \\ \mathrm{supp}(F) \subset B_g(p,\delta)}} \mu(M,g,F) \;=\; \mathrm{trace}\, \Box^{-1}_{S^n, V}.$$

Moreover, the limit is uniform over $p \in M$.

The topology and local geometry of M do not play any role in this local result. The general idea is that if the function F is supported close to a point, then since locally all manifolds look similar to $\mathbb{R}^n$, the degenerate metric $F^{2/n}g$ might as well be on the sphere. Indeed, $\mu(M,g,F)$ will be close to $\mathrm{trace}\, \Box^{-1}$ for some metric on the sphere. Moreover, by choosing the function F suitably we can ensure that the metric $F^{2/n}g$ blows up a neighborhood of a point to be approximately a round sphere, and $\mu(M,g,F)$ will be close to $\mathrm{trace}\, \Box^{-1}_{S^n,V}$. The details are given in Section 2.

Remark. In [DS2], a sequence of 2 dimensional tori of fixed area is constructed in such a way that $\mathrm{trace}\, \Delta^{-1}$ converges to the value for the round sphere. Whereas however, we make such a construction here for a fixed manifold by taking a sequence of conformal factors which concentrate at a point, in [DS2] the sequence of tori is constructed by taking conformal factors on a sequence of degenerating flat tori, and although the resulting tori approximate the sphere, the conformal factors do not concentrate.

Finally we show that the sharp form of the Logarithmic HLS inequality given in Theorem $1'$ always gives rise to a sharp form of the Beckner-Onofri inequality. It is well known on S^n that the logarithmic HLS inequalites and the Beckner inequalities are dual to each other, see [Be], [CL]. Those arguments can be extended to obtain a sharp Beckner-Onofri inequality in our situation. Suppose that M is a smooth manifold with measure $d\sigma$ with $d\sigma(M) = 1$, and $B : C^\infty(M) \to C^\infty(M)$ is a non-negative self-adjoint linear operator whose null space is equal to the constant functions, and which is invertible on the orthogonal complement of the constants.

Theorem 3. *Let α be a smooth function on M and let $\beta > 0$ be a constant. The following two statements are equivalent.*
(a). For $F \in C^\infty(M)$ with $F > 0$ and $\int_M F\, d\sigma = 1$,

$$\frac{\beta}{2}\int_M FB^{-2}F\, d\sigma \;\le\; \int_M F\log F\, d\sigma \;+\; \int_M F\alpha\, d\sigma.$$

(b). For $u \in C^\infty(M)$,

$$\frac{1}{2\beta}\int_M uB^2u\, d\sigma \;\ge\; \log\int_M e^{u-\alpha}\, d\sigma \;-\; \int_M u\, d\sigma.$$

Corollary 4. *(a). Suppose that the metric $g \in \Gamma$ attains the infimum on the left hand side of (1.7). Set $d\sigma = dV/V$. Then for $u \in L^2_{n/2}(M)$,*

$$\frac{V}{2\gamma_n}\int_M uAu\, d\sigma \;\ge\; \log\left(\int_M e^u\, d\sigma\right) \;-\; \int_M u\, d\sigma.$$

(b). Suppose that the inequality in (1.7) is an equality, and g is any metric in Γ. Set $d\sigma = dV/V$. Then for $u \in L^2_{n/2}(M)$,

$$\frac{V}{2\gamma_n}\int_M uAu\, d\sigma \;\ge\; \log\left(\int_M e^{u-\gamma_n m/2}\, d\sigma\right) \;-\; \int_M u\, d\sigma \;+\; \frac{\gamma_n}{2V}\,\mathrm{trace}\,\Box^{-1}_{S^n,V}.$$

Section 2. Proofs of the Theorems.

Proof that Theorem 1 and Theorem $1'$ are equivalent. In Theorem $1'$, a metric $g \in \Gamma$ is fixed, and other metrics in Γ are expressed in the form $F^{2/n}g$. The value of trace A^{-1} is expressed in terms of F by (1.5). There are only three things to check. Firstly, the fact that the metric $F^{2/n}g$ has constant Robin mass m is equivalent to (1.8). This follows from Lemma 2.1 below which is proved in [St1], [St2]. We give another proof in Section 3.

Lemma 2.1. Conformal change of the Robin mass. *Suppose g is a metric on M and F is a smooth positive function on M. Write $m_g(p)$ for the Robin constant of A_g at p. Then*

$$m_{F^{2/n}g}(p) \;=\; m_g(p) \;+\; \frac{2\log F}{\gamma_n} \;-\; \frac{2}{V_F}(A^{-1}F)(p) \;+\; \frac{1}{V_F^2}\int_M F(A^{-1}F)\, dV. \tag{2.1}$$

The second point that needs to be clarified is that in (1.4) we take an infimum over the set of smooth metrics Γ, while in (1.7) we allow degenerate metrics of the form $F^{2/n}g$ where the non-negative function F is in $(L\log L)^+(M)$. The fact that this does not change the infimum is due to the fact that the smooth positive functions are dense in $(L\log L)^+(M)$ and the functional $F \to \mu(M, g, F)$ is continuous on $(L\log L)^+(M)$. To see the latter, we need to show that $F \to \int_M FA^{-1}F\, dV$ is continuous, which follows easily from the next Lemma.

Lemma 2.2. A simple logarithmic Sobolev inequality.
For each $\varepsilon > 0$, there exists a constant $C_\varepsilon = C_\varepsilon(M, g)$ such that for all $F \in (L\log L)^+(M)$,

$$\|A^{-1}F(x)\|_\infty \le \ (1+\varepsilon)\frac{2}{\gamma_n}\int_M F\log F\,dV \ + \ C_\varepsilon\left(\int_M F\,dV + 1\right). \tag{2.2}$$

Remarks. In fact $A^{-1}F$ is continuous when $F \in (L\log L)^+(M)$. There is no bound of the form (2.2) when $\varepsilon = 0$.

This completes the proof that Theorem 1 and Theorem 1′ are equivalent, except we should just note that although (1.9) does not feature in Theorem 1, it follows easily from (1.8). Indeed, if $F = 1$ attains the minimum of the left hand side of (1.7), then from (1.8) we see that $m = m(x)$ must be constant. Then for $F \in (L\log L)^+(M)$ with $V_F = V$,

$$mV \ = \ \mu(M,g,1) \ \le \ \mu(M,g,F) \ = \ mV \ + \ \frac{2}{\gamma_n}\int_M F\log F\,dV \ - \ \frac{1}{V_F}\int_M FA^{-1}F\,dV.$$

Proof of Theorem 2. It will be convenient to give a characterization of trace$\,\Box^{-1}_{S^n,V}$ involving the flat space $\mathbb{R}^n$ rather than the sphere. This equivalence can be proved by using stereographic projection to identify $\mathbb{R}^n$ with S^n as in [CL]. We give a natural proof here using concentration arguments, see Proposition 2.6. Following previous notation, for a positive function f on $\mathbb{R}^n$ we set

$$V_f \ = \ \int_{\mathbb{R}^n} f\,dx.$$

Define

$$(L\log L)^+_c(\mathbb{R}^n) \ = \ \left\{f : \mathbb{R}^n \to [0,\infty) \ : \ f \text{ is measurable with compact support, } \int_{\mathbb{R}^n} f\log f\,dx < \infty\right\}.$$

and for $f \in (L\log L)^+_c(\mathbb{R}^n)$ define the functional on $\mathbb{R}^n$ analogous to $\mu(M,g,F)$:

$$\mu(\mathbb{R}^n, f) \ = \ \frac{2}{\gamma_n}\left(\int_{\mathbb{R}^n} f\log f\,dx \ - \ \frac{n}{V_f}\int_{\mathbb{R}^{2n}} f(x)\log|x-y|f(y)\,dxdy\right).$$

Set

$$\mu_V(\mathbb{R}^n) \ = \ \inf_{\substack{V_f = V \\ f\in(L\log L)^+_c(\mathbb{R}^n)}} \mu(\mathbb{R}^n, f).$$

We will now prove Theorem 2 with the quantity trace$\,\Box^{-1}_{S^n,V}$ replaced by $\mu_V(\mathbb{R}^n)$. It will then emerge that these quantities are equivalent.

Lemma 2.3. *For each point $p \in M$,*

$$\lim_{\delta\to 0} \ \inf_{\substack{V_F = V \\ F\in(L\log L)^+(M) \\ \mathrm{supp}(F)\subset B_g(p,\delta)}} \mu(M,g,F) \ = \ \mu_V(\mathbb{R}^n).$$

and the limit is uniform over $p \in M$.

Lemma 2.3 follows immediately from the next Lemma.

Lemma 2.4. *Given $\varepsilon > 0$, there exists $\delta > 0$ such that:*
(a). If F is supported in $B_g(p,\delta)$ for some $p \in M$, and $F \in (L\log L)^+(M)$, then there exists $f \in (L\log L)^+_c(\mathbb{R}^n)$ with $V_f = V_F$ such that

$$|\mu(M,g,F) \; - \; \mu(\mathbb{R}^n, f)| \; \leq \; \varepsilon V_F.$$

(b). If $f \in (L\log L)^+_c(\mathbb{R}^n)$ and $p \in M$, there exists $F \in (L\log L)^+(M)$ supported in $B_g(p,\delta)$ with $V_F = V_f$ such that

$$|\mu(M,g,F) \; - \; \mu(\mathbb{R}^n, f)| \; \leq \; \varepsilon V_f.$$

Notice that from (a) we get

$$\lim_{\delta\to 0} \inf_{\substack{V_F = V \\ F\in (L\log L)^+(M) \\ \mathrm{supp}(F)\subset B_g(p,\delta)}} \mu(M,g,F) \; \geq \; \mu_V(\mathbb{R}^n),$$

and from (b) we get

$$\lim_{\delta\to 0} \inf_{\substack{V_F = V \\ F\in (L\log L)^+(M) \\ \mathrm{supp}(F)\subset B_g(p,\delta)}} \mu(M,g,F) \; \leq \; \mu_V(\mathbb{R}^n).$$

Proof of Lemma 2.4. The idea of the proof is that if $F \in (L\log L)^+(M)$ is supported close to a point $p_0 \in M$, by taking suitable coordinates we can identify F with a function f on $\mathbb{R}^n$ so that $\mu(M,g,F)$ is close to $\mu(\mathbb{R}^n, f)$. Conversely, given $f \in (L\log L)^+_c(\mathbb{R}^n)$ we can rescale so that the support of f becomes small while V_f and $\mu(\mathbb{R}^n, f)$ remain constant, and then we can consider the function F on M which is given in coordinates by f. To carry out the details, for arbitrary fixed $\varepsilon > 0$, choose $\delta > 0$ so that if $d_g(p,q) < 2\delta$ then

$$\left| G(p,q) \; + \; \frac{2n}{\gamma_n}\log d_g(p,q) \; - \; m(p)\right| \; < \; \varepsilon. \tag{2.3}$$

We now choose good coordinates on M around each point p_0.

Lemma 2.5. *For each $\varepsilon > 0$, there exists $\delta > 0$ such that for each $p_0 \in M$ there exists an open neighborhood U of 0 in $\mathbb{R}^n$, and smooth coordinates $x = (x_1, \ldots, x_n) : B_g(p_0,\delta) \to U$ such that $x(p_0) = 0$,*

$$dV = dx = dx_1 \ldots dx_n, \tag{2.4}$$

and if $|x|$ is the Euclidean norm of $x \in \mathbb{R}^n$, then for $p, q \in B_g(p_0,\delta)$,

$$e^{-\varepsilon}|x(p) - x(q)| \; \leq \; d_g(p,q) \; \leq \; e^{\varepsilon}|x(p) - x(q)|. \tag{2.5}$$

This lemma is proved in Section 3.

Now given $\varepsilon > 0$ choose δ small enough so that (2.3) and the conclusion of Lemma 2.5 hold, and for $p_0 \in M$ take the coordinates $(x_1, \ldots, x_n)$ of Lemma 2.5. Then if $p, q \in B_g(p_0,\delta)$, we have

$$G(p,q) \; = \; -\frac{2n}{\gamma_n}\log|x(p) - x(q)| \; + \; m(x(p)) \; + \; \eta(p,q), \qquad |\eta(p,q)| < c\varepsilon, \qquad c = 1 + \frac{2n}{\gamma_n}.$$

If $F \in (L\log L)^+(M)$ is supported in $B(p_0,\delta)$, then

$$
\begin{aligned}
(2.6)\quad \frac{1}{V_F}\int_M FA^{-1}F\,dV \;&=\; \frac{1}{V_F}\int_M\int_M F(p)G(p,q)F(q)\,dV(p)dV(q)\\
&=\; -\frac{2n}{\gamma_n V_F}\int_M\int_M F(p)\log|x(p)-x(q)|F(q)\,dV(p)dV(q) \;+\; \int_M mF\,dV\\
&\qquad +\; \frac{1}{V_F}\int_M\int_M F(p)\eta(p,q)F(q)\,dV(p)dV(q).
\end{aligned}
$$

Define the function f on U by $f(x(p)) = F(p)$, and extend f to $\mathbb{R}^n$ by setting it equal to zero outside U. Then from (2.6),

$$
\begin{aligned}
(2.7)\quad \mu(M,g,F) \;&=\; \int_M mF\,dV \;+\; \frac{2}{\gamma_n}\int_M F\log F\,dV \;-\; \frac{1}{V_F}\int_M FA^{-1}F\,dV\\
&=\; \mu(\mathbb{R}^n,f) \;+\; \frac{1}{V_F}\int_M\int_M F(p)\eta(p,q)F(q)\,dV(p)dV(q).
\end{aligned}
$$

The second term on the right is bounded by $c\varepsilon V_F$. If we replace ε by ε/c we get (a). To prove (b), we note that the functional $\mu(\mathbb{R}^n,f)$ has a scale invariance. Indeed, for $f \in (L\log L)^+_c(\mathbb{R}^n)$ and $\lambda > 0$, we can set

$$
(2.8)\qquad h(x) \;=\; \frac{1}{\lambda^n}\, f\left(\frac{x}{\lambda}\right)
$$

and then one can check that $V_h = V_f$ and $\mu(\mathbb{R}^n,h) = \mu(\mathbb{R}^n,f)$. Now given $\varepsilon > 0$, we pick $\delta > 0$ so that (2.3) and the conclusion of Lemma 2.5 hold, and given $p_0 \in M$, choose the coordinates $(x_1,\ldots,x_n)$ on $B_g(p_0,\delta)$ from Lemma 2.5. Then the image of $B_g(p_0,\delta)$ under the coordinates map contains some ball $B_{\mathbb{R}^n}(0,\delta') = \{y \in \mathbb{R}^n : |y| < \delta'\}$. For each function $f \in (L\log L)^+_c(\mathbb{R}^n)$ we can choose λ sufficiently small so that $h(x)$ defined by (2.8) is supported in $B_{\mathbb{R}^n}(0,\delta')$. But then define the function F on M supported in $B_g(p_0,\delta)$ by $F(p) = h(x(p))$. Then (2.7) holds as before and we get (b).

To prove Theorem 2, it remains to show

Proposition 2.6.

$$
\mu_V(\mathbb{R}^n) \;=\; \mathrm{trace}\,\Box^{-1}_{S^n,V}.
$$

To prove this we first notice that by applying Lemma 2.3 to the sphere S^n with the round metric g_0 of volume V, we have

$$
(2.9)\qquad \mu_V(\mathbb{R}^n) \;=\; \lim_{\delta\to 0}\; \inf_{\substack{V_F=V\\ F\in(L\log L)^+(S^n)\\ \mathrm{supp}(F)\subset B_g(p,\delta)}} \mu(S^n,g_0,F)
$$

On the other hand, by Morpurgo's interpretation of the logarithmic HLS inequality,

$$
(2.10)\qquad \mathrm{trace}\,\Box^{-1}_{S^n,V} \;=\; \inf_{\substack{V_F=V\\ F\in(L\log L)^+(S^n)}} \mu(S^n,g_0,F).
$$

Since the infimum in (2.10) is over a bigger class of functions than in (2.9), we see that

$$
(2.11)\qquad \mathrm{trace}\,\Box^{-1}_{S^n,V} \;\le\; \mu_V(\mathbb{R}^n).
$$

The equality in (2.10) is obtained exactly when F is the Jacobian of a conformal transformation of the sphere. Now naturally there is no sequence of such functions F_j whose supports shrink down to a point. However, it is a well known that by combining stereographic projection with the scaling of Euclidean space, there exists a sequence of conformal transformations of S^n whose Jacobians F_j concentrate in the following sense.

Definition. Suppose that F_j is a sequence of positive functions in $L^1(M))$, we say that F_j *concentrates as* $j \to \infty$, if for every $\delta > 0$, there exists j_0 such that if $j \geq j_0$ then there exists $p_j \in M$ with

$$\int_{B_g(p_j,\delta)} F_j \, dV \; > \; V_{F_j}(1-\delta).$$

Proposition 2.6 follows from the next result which is an extension of Proposition 2.3. This then completes the proof of Theorem 2.

Proposition 2.7. *Let g_0 be the round metric of volume V on S^n. If $F_j \in (L\log L)^+(S^n)$ is a concentrating sequence with $V_{F_j} = V$, then*

$$\liminf_{j\to\infty} \mu(S^n, g, F_j) \; \geq \; \mu_V(\mathbb{R}^n).$$

Proposition 2.7 is proved in Section 3. In fact later on we will need the same result for general manifolds M, see Proposition 2.12.

Proof of Theorem 1′. To prove Theorem 1′ we will need to start with a weak form given in Proposition 2.9, and we will prove this by using the fact that it is true on S^n (or equivalently the fact that $\mu_V(\mathbb{R}^n)$ is finite) and moving this inequality over to M using a partition of unity. To accomplish this we need the following polarized form.

Lemma 2.8. *Let g be a metric on M with volume V, and let A be an operator on M of type $\Delta^{n/2}$. For a function F on M, set $\sigma_F = V_F/V$. The following three statements are equivalent.*
(a). For every $F \in (L\log L)^+(M)$ with $V_F = V$,

$$\frac{1}{V}\int_M F A^{-1} F \, dV \; \leq \; \frac{2}{\gamma_n}\int_M F \log F \, dV \; + \; C.$$

(b). For all $Q, R \in (L\log L)^+(\mathbb{R}^n)$ with $V_Q = V_R = V$,

$$\frac{1}{V}\int_M Q A^{-1} R \, dV \; \leq \; \frac{1}{\gamma_n}\left(\int_M Q \log Q \, dV + \int_M R \log R \, dV\right) \; + \; C.$$

(c). For all $Q, R \in (L\log L)^+(\mathbb{R}^n)$,

$$\frac{1}{V}\int_M Q A^{-1} R \, dV \; \leq \; \frac{1}{\gamma_n}\left(\sigma_R \int_M Q \log \frac{Q}{\sigma_Q} \, dV \; + \; \sigma_Q \int_M R \log \frac{R}{\sigma_R} \, dV\right) \; + \; C\,\sigma_Q\,\sigma_R.$$

Proposition 2.9. (**Weak logarithmic HLS inequality**.) *There exists a constant $C = C(M, g)$ such that if $F \in (L \log L)^+(M)$ with $V_F = V$, then*

$$\frac{2}{\gamma_n} \int_M F \log F \, dV \; - \; \frac{1}{V} \int_M F A^{-1} F \, dV \; \geq \; -C. \tag{2.12}$$

Lemma 2.8 and Proposition 2.9 are proved in Section 3. In order to prove Theorem 1′, we will need to get around a lack of compactness, which we do by adapting the ideas of [Y] to our situation. Let λ_1 be the lowest positive eigenvalue of A, and consider the functional

$$\mu^{(\varepsilon)}(M, g, F) \; = \; \int_M m F \, dV \; + \; \frac{2}{\gamma_n} \int_M F \log F \, dV \; - \; \frac{(1-\varepsilon)\lambda_1^{\varepsilon}}{V_F} \int_M F A^{-1-\varepsilon} F \, dV. \tag{2.13}$$

Proposition 2.10. *There exist a positive function $F^{(\varepsilon)} \in C^\infty(M)$ with $V_{F^{(\varepsilon)}} = V$ which minimizes $\mu^{(\varepsilon)}$, that is*

$$\mu^{(\varepsilon)}(M, g, F^{(\varepsilon)}) \; = \; \inf_{\substack{V_F = V \\ F \in (L \log L)^+(M)}} \mu^{(\varepsilon)}(M, g, F). \tag{2.14}$$

Moreover, $F^{(\varepsilon)}$ satisfies the equation

$$A^{1+\varepsilon}(\log F^{(\varepsilon)}) \; = \; \gamma_n \left(\frac{(1-\varepsilon)\lambda_1^{\varepsilon} F^{(\varepsilon)}(x)}{V} \; - \; \frac{A^{1+\varepsilon} m(x)}{2} \; - \; (1-\varepsilon)\lambda_1^{\varepsilon} \right). \tag{2.15}$$

Proof of Proposition 2.10. Applying Proposition 2.9, we see that if $V_F = V$, then

$$\begin{aligned} \mu^{(\varepsilon)}(M, g, F) \; &\geq \; \int_M m F \, dV \; + \; \frac{2}{\gamma_n} \int_M F \log F \, dV \; - \; \frac{1-\varepsilon}{V} \int_M F A^{-1} F \, dV \\ &= \; \frac{2\varepsilon}{\gamma_n} \int_M F \log F \, dV \; + \; (1-\varepsilon) \left(\frac{2}{\gamma_n} \int_M F \log F \, dV \; - \; \frac{1}{V} \int_M F A^{-1} F \, dV \right) \; + \; \int_M m F \, dV \\ &\geq \; \frac{2\varepsilon}{\gamma_n} \int_M F \log F \, dV \; - \; C' \end{aligned} \tag{2.16}$$

Choose a sequence F_j in $(L \log L)^+(M)$ with $V_{F_j} = V$ for all j and

$$\lim_{j \to \infty} \mu^{(\varepsilon)}(M, g, F_j) \; = \; \inf_{\substack{V_F = V \\ F \in (L \log L)^+(M)}} \mu^{(\varepsilon)}(M, g, F). \tag{2.17}$$

Then by (2.16), there exists C independent of j such that

$$\int_M F_j \log F_j \, dV \; \leq \; C.$$

Lemma 2.11. *Suppose that* $\Phi : [0, \infty) \to \mathbb{R}$ *is a continuous convex function with*

$$\frac{\Phi(t)}{t} \;\to\; \infty \qquad \text{as } t \to \infty.$$

Suppose that F_j *is a sequence of non-negative measurable functions on* M *such that*

$$\sup_j \int_M \Phi(F_j)\, dV \;=\; S \;<\; \infty.$$

Then after replacing F_j *by a subsequence, there exists* $F \in L^1(M)$ *such that* $F_j \to F$ *weakly, that is for every* $\phi \in C(M)$,

$$\int_M F_j \phi \, dV \;\to\; \int_M F \phi \, dV \qquad \text{as } j \to \infty,$$

and

$$\int_M \Phi(F)\, dV \;\le\; \liminf_{j\to\infty} \int_M \Phi(F_j)\, dV. \tag{2.18}$$

We will prove this Lemma in Section 3. Applying it to the function $\Phi(t) = t \log t$ and the sequence F_j in (2.17), we see that by taking a subsequence of F_j if necessary there exists $F^{(\varepsilon)} \in (L \log L)^+(M)$ with $F_j \to F^{(\varepsilon)}$ weakly in $L^1(M)$ and

$$\int_M F^{(\varepsilon)} \log F^{(\varepsilon)}\, dV \;\le\; \liminf_{j\to\infty} F_j \log F_j \, dV. \tag{2.19}$$

However, the integral kernel of $A^{-1-\varepsilon}$ is continuous on $M \times M$, and so by simple estimates $A^{-1-\varepsilon}$ is bounded from $L^1(M)$ to $C(M)$, and furthermore $\{A^{-1-\varepsilon}F : \|F\|_1 \le C\}$ is equicontinuous. Hence by taking a subsequence we can assume $A^{-1-\varepsilon}F_j$ converges in $C(M)$. But since F_j converges weakly in $L^1(M)$ to $F^{(\varepsilon)}$, this limit must equal $A^{-1-\varepsilon}F^{(\varepsilon)}$. Then

$$\lim_{j\to\infty} \int_M F_j A^{-1-\varepsilon} F_j \, dV \;=\; \int_M F^{(\varepsilon)} A^{-1-\varepsilon} F^{(\varepsilon)}\, dV.$$

But from this and (2.19) we get

$$\mu^{(\varepsilon)}(M, g, F^{(\varepsilon)}) \;\le\; \lim_{j\to\infty} \mu^{(\varepsilon)}(M, g, F_j),$$

and hence $\mu(M, g, F^{(\varepsilon)})$ satisfies (2.14). Now we need to show that $F^{(\varepsilon)}$ is positive and smooth. To simplify notation, write $F = F^{(\varepsilon)}$ and $B = (1-\varepsilon)\lambda_1^{\varepsilon} A^{-1-\varepsilon}$. We will first show that F is bounded below by a positive constant. For every bounded function H such that $F + H \in (L \log L)^+(M)$ and $\int_M H\, dV = 0$, we have

$$\begin{aligned} \mu^{(\varepsilon)}(M, g, F+H) - \mu^{(\varepsilon)}(M, g, F) \;&=\; \int_M mH\, dV \;+\; \frac{2}{\gamma_n} \int_M ((F+H)\log(F+H) - F\log F)\, dV \\ &\quad -\; \frac{2}{V}\int_M HBF\, dV \;-\; \frac{1}{V}\int_M HBH\, dV \end{aligned} \tag{2.20}$$

$$\leq \int_M mH\,dV \;+\; \frac{2}{\gamma_n}\int_M ((F+H)\log(F+H) - F\log F)\;dV \;-\; \frac{2}{V}\int_M HBF\,dV$$

(2.21)

$$\leq \frac{2}{\gamma_n}\int_M ((F+H)\log(F+H) - F\log F)\;dV \;+\; C\int_M |H|\,dV,$$

where

$$C \;=\; \|m\|_\infty \;+\; \frac{2}{V}\|BF\|_\infty.$$

We will show that if F is very small on a set of positive measure, then by choosing H appropriately, we can make (2.20) negative hence contradicting the fact that F minimizes $\mu^{(\varepsilon)}$. By the mean value theorem,

$$(f+h)\log(f+h) \;-\; f\log f \;<\; h(1+\log(f+h)) \;<\; 0$$

whenever

$$f>0,\ h>0,\quad f+h<1/e, \qquad\qquad \text{or} \qquad\qquad f>0,\ h<0,\ f+h>1/e.$$

Since the mean value of F equals 1, the set $U = \{p \in M : F(p) > 1/e\}$ has positive measure. Suppose that $N > 1$ and the set $U_N = \{p \in M : F(p) < e^{-N}\}$ has positive measure. Define H so that $\int_M H\,dV = 0$ and

$$\begin{cases} 0 < H < e^{-N} - F & \text{on } U_N \\ e^{-1} - F < H < 0 & \text{on } U \\ H = 0 & \text{on } M \setminus (U_N \cup U). \end{cases}$$

Then by (2.21),

$$\int_M ((F+H)\log(F+H) - F\log F)\;dV \;\leq\; -N\int_{U_N} H\,dV \;=\; \frac{-N}{2}\int_M |H|\,dV.$$

From (2.21), we see that if $N/\gamma_n > C$ then $\mu^{(\varepsilon)}(M,g,F+H) - \mu^{(\varepsilon)}(M,g,F) < 0$, which contradicts F being a local minimum. Hence F is bounded below. A similar argument shows that F is bounded above. Setting the variation of (2.20) about $H = 0$ equal to zero, shows that

$$m \;+\; \frac{2\log F}{\gamma_n} \;-\; \frac{2BF}{V} \;=\; \text{constant.} \tag{2.22}$$

However, since $\log F$ is in $L^\infty(M)$ and m is smooth, we can apply elliptic regularity theory to conclude that F is smooth. Applying the operator $A^{-1-\varepsilon}$ to (2.22) gives (2.15) and completes the proof of Proposition 2.10.

We return to the proof of Theorem 1′. We want to obtain a minimizer for $\mu(M,g,F)$. Take a sequence $\varepsilon_j \searrow 0$, and consider the sequence of functions $F^{(\varepsilon_j)}$. Since $F^{(\varepsilon_j)}$ is minimizing for $\mu^{(\varepsilon)}(M,g,F)$ and $\mu^{(\varepsilon_j)}(M,g,F)$ decreases to $\mu(M,g,F)$ as $\varepsilon_j \searrow 0$, we see that

$$\lim_{j\to\infty} \mu(M,g,F^{(\varepsilon_j)}) \;=\; \inf_{\substack{V_F = V \\ F \in (L\log L)^+(M)}} \mu(M,g,F). \tag{2.23}$$

We will need the analog of Proposition 2.7 on the manifold M.

Proposition 2.12. *If $F_j \in (L \log L)^+(M)$ is a concentrating sequence with $V_{F_j} = V$, then*

$$\liminf_{j\to\infty} \mu(M, g, F_j) \;\geq\; \mu_V(\mathbb{R}^n). \tag{2.24}$$

Proposition 2.12 will be proved in Section 3. Applying Propositions 2.12 and 2.6 to the sequence $F^{(\varepsilon_j)}$, we see that if $F^{(\varepsilon_j)}$ concentrates, then

$$\inf_{\substack{V_F=V \\ F\in(L\log L)^+(M)}} \mu(M,g,F) \;=\; \mu_V(\mathbb{R}^n) \;=\; \operatorname{trace} \square^{-1}_{S^n,V}.$$

On the other hand if the sequence $F^{(\varepsilon_j)}$ does not concentrate then after taking a subsequence, there exists $\delta > 0$ such that for every $p \in M$ and every j,

$$\int_{B(p,\delta)} F^{(\varepsilon_j)}\, dV \;\leq\; V(1-\delta).$$

We can apply the following result which is proved in Section 3.

Proposition 2.13. *(***Improved logarithmic HLS inequality for non-concentrating functions***.) Given $\delta > 0$, there exists a constant $C = C(M, g, \delta)$ such that if $F \in (L \log L)^+(M)$ satisfies $V_F = V$ and is such that for every $p \in M$,*

$$\int_{B_g(p,\delta)} F\, dV \;\leq\; (1-\delta)V_F,$$

then

$$(1-\delta)\frac{2}{\gamma_n}\int_M F\log F\, dV \;-\; \frac{1}{V_F}\int_M FA^{-1}F\, dV \;\geq\; -C.$$

From this result and the fact that $\mu(M, g, F^{(\varepsilon_j)})$ is bounded above, we see that there is a uniform bound

$$\int_M F^{(\varepsilon_j)} \log F^{(\varepsilon_j)}\, dV \;\leq\; C, \tag{2.25}$$

Lemma 2.14. *The operators $A^{-\varepsilon}$ with $\varepsilon \in [0, 1/2]$ are uniformly bounded on $C^k(M)$.*

We sketch the proof of Lemma 2.14 in Section 3. Applying Lemmas 2.2 and 2.14 to (2.25), we get a constant C such that

$$\|A^{-1-\varepsilon_j} F^{(\varepsilon_j)}\|_\infty \;\leq\; C$$

From (2.15), this gives a uniform bound on $\|\log F^{(\varepsilon)}\|_\infty$. Indeed, since the null space of A is the constant functions, we get a bound of the form

$$\|\log F^{(\varepsilon)} - c_\varepsilon\|_\infty \;\leq\; C',$$

where C' is a constant independent of ε but c_ε is an unknown constant which may depend on ε. However, since the average value of $F^{(\varepsilon)}$ is 1, we see that $F^{(\varepsilon)}$ takes the value 1 and so $\log F^{(\varepsilon)}$ takes the value

zero. Hence $|c_\varepsilon| \le C'$. Hence we get uniform bounds on $\|F^{(\varepsilon_j)}\|_\infty$. However, A^{-1} is bounded from C^k to C^{k+n-1} and so applying (2.15) again, we obtain uniform bounds on $\|F^{(\varepsilon_j)}\|_{C^{n-1}(M)}$. Continuing in this way, for all k we get bounds on $\|F^{(\varepsilon_j)}\|_{C^k(M)}$ which are uniform in j. Hence using the Azela-Ascoli theorem and and a diagonalization argument, we can find a subsequence which converges in $C^\infty(M)$ to a smooth positive function F which attains the right hand side of (2.23), and which satisfies the limiting equation (1.8).

Proof of Theorem 3. Set $\sigma_u = \int_M u\, d\sigma$. Our starting point is the inequality

$$\begin{aligned} \int_M F(u-\sigma_u)\, d\sigma \ &\le\ \left(\int_M |B^{-1}F|^2\, d\sigma\right)^{1/2} \left(\int_M |Bu|^2\, d\sigma\right)^{1/2} \\ &\le\ \frac{\beta}{2}\int_M FB^{-2}F\, d\sigma \ +\ \frac{1}{2\beta}\int_M uB^2u\, d\sigma, \end{aligned} \tag{2.26}$$

with equality if $F = c + B^2u/\beta$, where c is a constant.
(a)$\Rightarrow$(b). We follow [CL]. Assume that (a) holds. Then for $u \in C^\infty(M)$, applying (2.26) with $F > 0$ and $\int_M F\, d\sigma = 1$, we have

$$\begin{aligned} \frac{1}{2\beta}\int_M uB^2u\, d\sigma \ +\ \int_M u\, d\sigma \ &\ge\ \int_M Fu\, d\sigma \ -\ \frac{\beta}{2}\int_M FB^{-2}F\, d\sigma \\ &\ge\ \int_M F(u-\alpha)\, d\sigma \ -\ \int_M F\log F\, d\sigma. \end{aligned}$$

We get (b) by choosing

$$F \ =\ \frac{e^{u-\alpha}}{\int_M e^{u-\alpha}\, d\sigma}.$$

(b)$\Rightarrow$(a). Assume that (b) holds, and suppose $F \in C^\infty(M)$ satisfies $F > 0$ and $\int_M F\, d\sigma = 1$, and set $u = \beta B^{-2}F$. Then

$$\begin{aligned} \frac{\beta}{2}\int_M FB^{-2}F\, d\sigma \ &=\ \int_M Fu\, d\sigma \ -\ \frac{1}{2\beta}\int_M uB^2u\, d\sigma \\ &\le\ \int_M Fu\, d\sigma \ -\ \log\int_M e^{u-\alpha}\, d\sigma \\ &\le\ \int_M F\log F\, d\sigma \ +\ \int_M F\alpha\, d\sigma, \end{aligned}$$

where the last line follows from Jensen's inequality:

$$\int_M (u-\alpha-\log F)F\, d\sigma \ \le\ \log\int_M e^{u-\alpha-\log F}F\, d\sigma \ =\ \log\int_M e^{u-\alpha}\, d\sigma.$$

Remark. In proving (a)$\Rightarrow$(b), the choice $F = e^{u-\alpha}/\int_M e^{u-\alpha}\, d\sigma$ is the Legendre function for the functional $\int_M F\log F\, d\sigma$.

Section 3. Proofs of the auxiliary results.

Proof of Lemma 2.1. Write $d(p,q)$ for the distance from p to q in the metric g. The Green's function $G(p,q)$ for $A = A_g$ is a smooth function of $(p,q) \in M \times M$ away from the diagonal p,q, and is characterized by the following conditions:

$$
\begin{aligned}
& A_q G(p,q) = -\frac{1}{V} && \text{for } q \neq p, \\
(3.1)\qquad & G(p,q) \; + \; \frac{2n}{\gamma_n} \log d(p,q) && \text{is bounded}, \\
(3.2)\qquad & \int_M G(p,q)\, dV(q) \; = \; 0, && \\
& G(p,q) = G(q,p). &&
\end{aligned}
$$

Here, $A_q G(p,q)$ denotes the operator A applied to $G(p,q)$ in the variable q. Consider the function

$$E(p,r,q) = G(p,q) - G(r,q)$$

which satisfies

$$
\begin{aligned}
(3.3)\qquad & A_q E(p,r,q) \; = \; 0, \qquad \text{for } q \neq p,r, \\
& E(p,r,q) \; + \; \frac{2n}{\gamma_n} (\log d(p,q) - \log d(r,q)) \quad \text{is bounded}, \\
& \int_M E(p,r,q)\, dV(q) = 0.
\end{aligned}
$$

Now set $\tilde{g} = F^{2/n} g$. We write $\tilde{G}$ for the Green's function for $\tilde{g}$, and

$$\tilde{E}(p,r,q) = \tilde{G}(p,q) - \tilde{G}(r,q).$$

Because of (3.3) we see that

$$A_q(\tilde{E}(p,r,q) - E(p,r,q)) = 0.$$

By (3.1) we see that $\tilde{E}(p,r,q) - E(p,r,q)$ is bounded on M. We conclude that $\tilde{E}(p,r,q) - E(p,r,q)$ is constant. We can compute this constant by applying (3.2) for $\tilde{E}$, and we get

$$\tilde{G}(p,q) - \tilde{G}(r,q) \; = \; G(p,q) - G(r,q) \; - \; \frac{1}{V_F}(A^{-1}F)(p) \; + \; \frac{1}{V_F}(A^{-1}F)(r).$$

Averaging with respect to $F(r) dV(r)$, we see that

$$\tilde{G}(p,q) \; = \; G(p,q) \; - \; \frac{1}{V_F}(A^{-1}F)(q) \; - \; \frac{1}{V_F}(A^{-1}F)(p) \; + \; \frac{1}{V_F^2} \int_M F A^{-1} F \, dV$$

Now

$$m(p) \; = \; \lim_{q \to p} \left(G(p,q) + \frac{2n}{\gamma_n} \log d(p,q) \right),$$

writing $\tilde{d}$ for the distance function for the metric $\tilde{g}$, we see from (3.12) that

$$m_{F^{2/n}g}(p) \;=\; \lim_{q\to p}\bigg(\frac{2n}{\gamma_n}\log\tilde{d}(p,q) \;-\; \frac{2n}{\gamma_n}\log d(p,q) \;+\; m_g(p) \\ \;-\; \frac{1}{V_F}(A^{-1}F)(q) \;-\; \frac{1}{V_F}(A^{-1}F)(p) \;+\; \frac{1}{V_F^2}\int_M FA^{-1}F\,dV,\bigg)$$

so

$$m_{F^{2/n}g}(p) \;=\; m_g(p) \;+\; \frac{2\log F(p)}{\gamma_n} \;-\; \frac{2}{V_F}A^{-1}F(p) \;+\; \frac{1}{V_F^2}\int_M FA^{-1}F\,dV.$$

Hence defining

$$\operatorname{trace} A_g^{-1} \;=\; \int_M m\,dV,$$

we have

$$\operatorname{trace} A_{F^{2/n}g}^{-1} \;=\; \int_M mF\,dV \;+\; \frac{2}{\gamma_n}\int_M F\log F\,dV \;-\; \frac{1}{V_F}\int_M F(A^{-1}F)\,dV.$$

Proof of Lemma 2.2. Writing $G(p,q)$ for the Green's function for A_g, we have

$$G(p,q) \;=\; -\frac{2n}{\gamma_n}\log d(p,q) \;+\; E(p,q),$$

where $E(p,q)$ is bounded. Hence

$$A^{-1}F(p) \;=\; \int_M G(p,q)F(q)\,dV(q) \;=\; \frac{-2n}{\gamma_n}\int_M F(q)\log d(p,q)\,dV(y) \;+\; \int_M E(p,q)F(q)\,dV(q),$$

and the second term on the right is bounded by $V\sup E$, which can be absorbed in C_ε. For the first term, taking $0<\delta<n$, we have

$$\begin{aligned} n\int_M F(q)\log d(p,q)\,dV(q) \;&\le\; \int_{\log F>-(n-\delta)\log d(p,q)} F(q)(-n\log d(p,q))\,dV(q) \\ &\qquad +\; \int_{\log F\le-(n-\delta)\log d(p,q)} F(q)(-n\log d(p,q))\,dV(q) \\ &\le\; \frac{n}{n-\delta}\int_M F\log F\,dV \;+\; n\int_M (d(p,q))^{\delta-n}\log d(p,q)\,dV(q). \end{aligned}$$

The second integral on the right converges, so choosing δ small enough so $n/(n-\delta)<1+\varepsilon$, we get (2.2).

Proof of Lemma 2.5. For each point $p_0\in M$ we there exist smooth coordinates $(y_1,\ldots,y_n)$ mapping some open neighborhood of p_0 diffeomorphically onto an open ball $B_{\mathbb{R}^n}(0,r)$. Then writing

$$g_{ij} = g\left(\partial_{y_i},\partial_{y_j}\right),$$

we have

$$dV \;=\; \sqrt{|\det g_{ij}|}\;dy_1\ldots dy_n.$$

Set

$$x_i = y_i, \quad i < n, \qquad\qquad x_n = \int_0^{y_n} \sqrt{|\det g(y_1, \ldots, y_{n-1}, t)|}\, dt.$$

Then by the inverse function theorem, the map $y \to x$ is a diffeomorphism from some neighborhood of zero to a neighborhood of zero, so $p \to x(y(p))$ defines smooth coordinates on some neighborhood of p_0. Moreover, by computing the Jacobian $|\partial y/\partial x|$ we see that

$$dV \; = \; dx_1 \ldots dx_n.$$

We now work in the coordinates x and write

$$g_{ij} = g(\partial_{x_i}, \partial_{x_j}).$$

By applying a linear transformation to $(x_1, \ldots, x_n)$ if necessary we can assume that at $x = 0$, $g_{ij} = \delta_{ij}$. Then there exists $r > 0$ such that for $|x| < r$

$$\left(\sum_{i,j} |g_{ij} - \delta_{ij}|^2 \right)^{1/2} \; < \; 1 - e^{-\varepsilon}.$$

But then for $v = (v_1, \ldots, v_n)$,

$$e^{-\varepsilon}|v| \; \leq \; g(v,v)^{1/2} \; \leq \; e^{\varepsilon}|v|,$$

and for any curve γ in the coordinate ball $\{x : |x| < r\}$, we have that if $|\gamma|$ is the Euclidean length of γ and $L_g(\gamma)$ is the length of γ in the metric g, then

$$e^{-\varepsilon}|\gamma| \; \leq \; L_g(\gamma) \; \leq \; e^{\varepsilon}|\gamma|.$$

But then minimizing the middle term or the term on the right over curves γ joining y to z gives (2.5). Now the ball $|x| < r$ contains some geodesic ball $B_g(p_0, \delta)$.

We can choose δ independent of p_0 by a simple compactness argument. Indeed, by compactness there is a finite cover of balls $B_g(p_1, \delta_1)$, $\ldots$, $B_g(p_N, \delta_N)$ on which we have coordinates satisfying (2.4) and (2.5). But since this is an open cover, we can find $\delta > 0$ such that for each $p_0 \in M$, $B_g(p_0, \delta) \subset B_g(p_j, \delta_j)$ for some $j \in \{1, 2, \ldots, N\}$.

Proof of Proposition 2.7. This is a special case of Proposition 2.12 proved below.

Proof of Lemma 2.8 (a)$\Rightarrow$(b). Just apply Cauchy-Schwarz to get.

$$\begin{aligned} \frac{1}{V}\int_M QA^{-1}R\, dV \; &\leq \; \left(\frac{1}{V}\int_M QA^{-1}Q\, dV\right)^{1/2} \left(\frac{1}{V}\int_M RA^{-1}R\, dV\right)^{1/2} \\ &\leq \; \frac{1}{2}\left(\frac{1}{V}\int_M QA^{-1}Q\, dV \; + \; \frac{1}{V}\int_M RA^{-1}R\, dV\right), \end{aligned}$$

and then apply (a).

(b)$\Rightarrow$(c). Just apply (b) to Q/σ_Q and R/σ_R.

(c)$\Rightarrow$(a). (a) is just a special case of (c) when $Q = R = F$ and $V_F = V$.

Proof of Proposition 2.9. Now $\mu_V(\mathbb{R}^n)$ is bounded below (see [CL] or (2.11)). Moreover, applying Lemma 2.4(a) with $\varepsilon = 1$, we see that there exists $\delta > 0$ such that if and $p \in M$ and $F \in (L \log L)^+(M)$ is supported in $B_g(p, \delta)$, then

$$\mu(M, g, F) \;\geq\; \mu_{V_F}(\mathbb{R}^n) \;-\; V_F.$$

Hence from the definition of $\mu(M, g, F)$ and the fact that m is bounded, we get a constant C such that if $F \in (L \log L)^+(M)$ is supported in $B_g(p, \delta)$ with $V_F = V$, then

$$\frac{1}{V} \int_M F A^{-1} F \, dV \;\leq\; \frac{2}{\gamma_n} \left(\int_M F \log F \, dV \right) \;+\; C.$$

This proves (2.12) when F is supported in $B_g(p, \delta)$ for some $p \in M$, but we want to remove this restriction on the support of F. As in Lemma 2.8, we see that whenever $Q, R \in (L \log L)^+(\mathbb{R}^n)$ with $Q, R \in B_g(p, \delta)$ for some p, then

$$\frac{1}{V} \int_M Q A^{-1} R \, dV \;\leq\; \frac{1}{\gamma_n} \left(\sigma_R \int_M Q \log \frac{Q}{\sigma_Q} \, dV \;+\; \sigma_Q \int_M R \log \frac{R}{\sigma_R} \, dV \right) \;+\; C \, \sigma_Q \, \sigma_R.$$

Choose closed sets $W_1, \ldots, W_N$ which cover M such that the measure of $W_i \cap W_j$ equals zero if $i \neq j$. Suppose also that the sets W_j are sufficiently small that if $W_i \cap W_j \neq \emptyset$ then there exists p with W_i and W_j contained in $B_g(p, \delta)$. We can choose $\varepsilon > 0$ such that if $W_i \cap W_j = \emptyset$ then the distance from W_i to W_j is at least ε. Let χ_j denote the characteristic function of W_j. For $F \in (L \log L)^+(M)$, set $F_j = \chi_j F$. Set

$$C_1 \;=\; \sup_{d(p,q) > \varepsilon} |G(p, q)|.$$

Then

$$\int_M F A^{-1} F \, dV \;=\; \sum_{W_i \cap W_j = \emptyset} \int_M F_i A^{-1} F_j \, dV \;+\; \sum_{W_i \cap W_j \neq \emptyset} \int_M F_i A^{-1} F_j \, dV.$$

The first sum on the right is bounded by $C_1 V_F^2$. The second sum on the right is bounded by

$$\sum_{W_i \cap W_j \neq \emptyset} \left(\frac{V_{F_i}}{\gamma_n} \int_M F_j \log \left(\frac{F_j}{\sigma_{F_j}} \right) dV \;+\; \frac{V_{F_j}}{\gamma_n} \int_M F_i \log \left(\frac{F_i}{\sigma_{F_i}} \right) dV \;+\; \frac{C V_{F_i} V_{F_j}}{V} \right). \tag{3.4}$$

However, since the function $t \to t \log t$ is convex, Jensen's inequality gives

$$\int_M F_j \log \left(\frac{F_j}{\sigma_{F_j}} \right) dV \;\geq\; 0.$$

Hence assuming $V_F = V$, (3.4) is bounded above by

$$\begin{aligned}
&\sum_{i,j} \left(\frac{V_{F_i}}{\gamma_n} \int_M F_j \log \left(\frac{F_j}{\sigma_{F_j}} \right) dV \;+\; \frac{V_{F_j}}{\gamma_n} \int_M F_i \log \left(\frac{F_i}{\sigma_{F_i}} \right) dV \;+\; \frac{C V_{F_i} V_{F_j}}{V} \right) \\
&= \frac{2 V_F}{\gamma_n} \sum_j \int_M F_j \log \left(\frac{F_j}{\sigma_{F_j}} \right) dV \;+\; \frac{C V_F^2}{V} \\
&= \frac{2V}{\gamma_n} \int_M F \log F \, dV \;-\; \frac{2V}{\gamma_n} \sum_j V_{F_j} \log V_{F_j} \;+\; C'
\end{aligned}$$

where C' depends only on C and V. Setting $C'' = -\min_{t>0} t \log t$, we get

$$\int_M F A^{-1} F \, dV \ \leq \ \frac{2V}{\gamma_n} \int_M F \log F \, dV \ + \ \frac{2NC''V}{\gamma_n} \ + \ C' \ + \ C_1 V^2.$$

This completes the proof of Proposition 2.9.

Proof of Lemma 2.11. Since the function $\Phi(t)$ grows faster at infinity than the function t, we see that functions F_j are uniformly bounded in $L^1(M)$. Hence by taking a subsequence we can assume F_j converges weakly to a measure $d\sigma$, that is for all $\phi \in C(M)$,

$$\int_M F_j \phi \, dV \ \to \ \int_M \phi \, d\sigma \qquad \text{as } j \to \infty. \tag{3.5}$$

But applying Tchebychev's inequality to the functions F_j, we see that for a measurable set $U \subset M$,

$$\int_U F_j \, dV \ \leq \ \int_{\{x \in U : F_j(x) \leq \lambda\}} F_j \, dV \ + \ \int_{\{x \in U : F_j(x) > \lambda\}} F_j \, dV \ \leq \ \lambda \int_U \, dV \ + \ \frac{S}{\Phi(\lambda)},$$

and this shows that the limit $d\sigma$ is absolutely continuous with respect to the measure dV, and hence equals $F \, dV$ for some function $F \in L^1(M)$, and (3.5) holds when ϕ is the characteristic function of a measurable set. Now for $0 \leq m \leq N^2$,

$$I_N^m \ = \ \left\{ x : \frac{m}{N} \ \leq \ F \ \leq \ \frac{m+1}{N} \right\}, \qquad\qquad V_N^m = \int_{I_N^m} dV.$$

Then by Jensen's inequality,

$$\begin{aligned} \sum_{m=0}^{N^2} \Phi\left(\frac{1}{V_N^m} \int_{I_N^m} F \, dV \right) \ V_N^m \ &= \ \liminf_{j \to \infty} \sum_{m=0}^{N^2} \Phi\left(\frac{1}{V_N^m} \int_{I_N^m} F_j \, dV \right) \ V_N^m \\ &\leq \ \liminf_{j \to \infty} \sum_{m=0}^{N^2} \int_{I_N^m} \Phi(F_j) \, dV \ = \ \liminf_{j \to \infty} \int_{\{F \leq N^2\}} \Phi(F_j) \, dV \end{aligned}$$

However, this gives (2.18), because as $N \to \infty$ the left hand side of (3.8) converges to the left hand side of (2.18) and the right hand side of (3.8) converges to the right hand side of (2.18).

Proof of Proposition 2.12. From Lemma 2.3 we see that (2.24) holds if the supports of the functions F_j are shrinking to a point. In order to prove equality for general concentrating sequences, take $\delta_j \to 0$ with

$$\int_{B_g(p_j, \delta_j)} F_j \, dV \ > \ V(1 - \delta_j).$$

Let χ be the characteristic function of $B(p_j, \delta_j)$ and set

$$Q_j = \chi \, F_j, \qquad\qquad R_j = F_j - Q_j.$$

Now we apply Lemma 2.4. Since $\delta_j \to 0$ and Q_j is supported in $B(p_j, \delta_j)$, we can find a sequence $\varepsilon_j > 0$ with $\varepsilon_j \to 0$ such that

$$\mu(M, g, Q_j) \;\geq\; \mu_{V_{Q_j}}(\mathbb{R}^n) \;-\; \varepsilon_j V_{Q_j}. \tag{3.6}$$

To simplify the notation, we fix j and set $F = F_j$, $Q = Q_j$, $R = R_j$. Then by Proposition 2.9 and Lemma 2.8, we have

$$\int_M QA^{-1}R\,dV \;\leq\; \frac{1}{\gamma_n}\left(V_R\int_M Q\log\frac{Q}{\sigma_Q}\,dV \;+\; V_Q\int_M R\log\frac{R}{\sigma_R}\,dV\right) \;+\; C\,V_Q\,V_R. \tag{3.7}$$

But since Q and R have disjoint supports, one easily checks that

$$\begin{aligned}
V\;\mu(M,g,F) \;&=\; V\int_M Fm\,dV \;+\; \frac{2V_F}{\gamma_n}\int_M F\log F\,dV \;-\; \int_M FA^{-1}F\,dV\\
&=\; (V_Q+V_R)\int_M (Q+R)m\,dV \;+\; \frac{2(V_Q+V_R)}{\gamma_n}\int_M (Q\log Q \;+\; R\log R)\;dV\\
&\qquad\qquad -\; \int_M (Q+R)A^{-1}(Q+R)\,dV\\
&=\; V_Q\;\mu(M,g,Q) \;+\; \frac{2V_R}{\gamma_n}\int_M R\log R\,dV \;-\; \int_M RA^{-1}R\,dV\\
&\qquad +\; \frac{2V_Q}{\gamma_n}\int_M R\log R\,dV \;+\; \frac{2V_R}{\gamma_n}\int_M Q\log Q\,dV \;-\; 2\int_M QA^{-1}R\,dV\\
&\qquad +\; \int_M (V_QR+V_RF)m\,dV.
\end{aligned}$$

Then applying (3.7), we get

$$\begin{aligned}
V\;\mu(M,g,F) \;&\geq\; V_Q\;\mu(M,g,Q) \;+\; \frac{1}{\gamma_n}\Big(2V_R^2\log\sigma_R \;+\; 2V_RV_Q(\log\sigma_R+\log\sigma_Q)\Big)\\
&\qquad -\; \frac{CV_R^2}{V} \;-\; \frac{2CV_RV_Q}{V} \;-\; V_R(V_Q+V)\max_M|m|\\
&\geq\; V_Q\;\mu(M,g,Q) \;+\; 2(V_R\log V_R)V_F \;-\; C'V_R,
\end{aligned}$$

where C' depends only on (M,g). Allowing j to vary and applying (3.6), we have

$$\mu(M,g,F_j) \;\geq\; \frac{V_{Q_j}}{V}\left(\mu_{V_{Q_j}}(\mathbb{R}^n) \;-\; \varepsilon_j V_{Q_j}\right) \;+\; 2(V_{R_j}\log V_{R_j}) \;-\; C'V_R.$$

But the right hand side converges to $\mu_V(\mathbb{R}^n)$ as $j \to \infty$. This completes the proof of Proposition 2.12.

Proof of Proposition 2.13. We modify the proof of Proposition 2.9. (Write δ' for the value of δ appearing in the proof of Proposition 2.9 to distinguish it from the value δ in the statement of Proposition 2.13.) When we choose the sets $U_1,\ldots,U_N$ we place the additional restriction that $U_j \subset B_g(p_j,\delta/2)$ for some points $p_j \in M$. Then choosing V_j as above, we get that for fixed j,

$$\sum_{\{i:V_i\cap V_j\neq\emptyset\}} V_{F_i} \;\leq\; \int_{B_g(p_j,\delta)} F\,dV \;\leq\; (1-\delta)V_F. \tag{3.8}$$

Using this to bound (3.4), we get

$$\int_M FA^{-1}F\,dV \;\le\; \frac{2(1-\delta)V_F}{\gamma_n}\sum_j \int_M F_j \log\left(\frac{F_j}{\sigma_{F_j}}\right) dV \;+\; \frac{CV}{V} \;+\; C_1V^2$$

$$\le\; \frac{2(1-\delta)V}{\gamma_n}\int_M F\log F\,dV \;+\; C_2,$$

where C_2 depends only on (M,g) and δ. This completes the proof of Proposition 2.13.

Proof of Lemma 2.14. We will assume that the reader is familiar with the standard theory of elliptic pseudodifferential operators, in particular the construction of powers, see for example [Se]. We work with the spaces $\Psi\mathrm{DO}^m(M)$ of classical pseudodifferential operators of order $m \in \mathbb{R}$. This is a class of operators on $C^\infty(M)$. For local coordinates on $\Omega \subset M$, an operator $B \in \Psi\mathrm{DO}^m(M)$ acts on smooth functions F supported in Ω by

$$BF(x) \;=\; \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_\Omega e^{i(x-y)\cdot\xi}\, b(x,\xi)F(y)\,dyd\xi,$$

where x, y are the coordinates of points in Ω and where the symbol $b(x,\xi)$ satisfies estimates

$$\left|\partial_x^\alpha \partial_\xi^\beta b(x,\xi)\right| \;\le\; C_{\alpha\beta}(1+|\xi|)^{m-|\beta|}.$$

We say that a set of operators $B^{(\varepsilon)}$ is *uniformly bounded in $\Psi DO^m(M)$* if the constants $C_{\alpha\beta}$ can be chosen independent of ε. If $m < 0$ then B is an integral operator. It's Schwartz kernel $K(p,q)$ is a function on $M \times M$ which is given in local coordinates on $\Omega \times \Omega$ by

$$K(x,y) \;=\; \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\, b(x,\xi)\,d\xi.$$

For $r > 0$, the operators $A^{-\varepsilon}$ with $\varepsilon \in [0,r]$ are uniformly bounded in $\Psi\mathrm{DO}^0(M)$. The following standard Lemma is not quite sufficient to prove Lemma 2.14.

Lemma 3.1. *If $B^{(\varepsilon)}$ are uniformly bounded in $\Psi DO^m(M)$ and $m < k$, then the operators $B^{(\varepsilon)}$ are uniformly bounded from $C^k(M)$ to $C(M)$.*

We start by proving Lemma 2.14 in the case $k = 0$. We want to apply Lemma 2.14 to A^ε, but we will need to deal separately with the principal symbol. To do this we will apply the following trivial lemma.

Lemma 3.2. *If for $F \in C(M)$ we set*

$$BF(p) \;=\; \int_M K(p,q)F(q)\,dV(q),$$

where

$$\sup_p \int_M |K(p,q)|\,dV(q) \;\le\; C,$$

then B is bounded on $C(M)$ and

$$\|BF\|_\infty \ \leq \ C\|F\|_\infty.$$

Working in coordinates $(x_1, \ldots, x_n)$, we write g^{ij} for the components of g^{-1}. Then the principal symbol of $A^{-\varepsilon}$ is $(g(x,\xi))^{-n\varepsilon/2}$ where

$$g(x,\xi) \ = \ \sum_{i,j} g^{ij}\xi_i\xi_j.$$

Working in normal coordinates about the point p_0 then this symbol is just $|\xi|^{-n\varepsilon}$. The kernel of the operator corresponding to this symbol is given by taking the inverse Fourier transform of $|\xi|^{-n\varepsilon}$:

$$\text{(3.9)} \qquad \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}|\xi|^{-n\varepsilon}\,d\xi \ = \ C(\varepsilon)\,|x|^{\varepsilon-n} \qquad\qquad C(\varepsilon) \ = \ (2\pi)^{-n/2}2^{n(\frac{1}{2}-\varepsilon)}\frac{\Gamma((n(1-\varepsilon)/2)}{\Gamma(\varepsilon/2)}.$$

Notice that $C(\varepsilon)/\varepsilon$ is bounded as $\varepsilon \to 0$. Here, x are the normal coordinates on M centered at p_0, so $|x|$ measures the distance from p_0. The upshot of this is that we can write

$$A^{-\varepsilon} \ = \ A_0^{-\varepsilon} \ + \ B^{(\varepsilon)},$$

where

$$\text{(3.10)} \qquad A_0^{-\varepsilon}F(p) \ = \ C(\varepsilon)\int_M (d(p,q))^{\varepsilon-n}\phi(d(p,q))F(q)\,dV(q).$$

Here ϕ is a smooth cut off function, and the operators $B^{(\varepsilon)}$ are uniformly bounded in the class $\Psi\mathrm{DO}^{-1}(M)$. (There is a technical point here: we are writing q in normal coordinates around p instead of using a fixed coordinates chart for both p and q.)

Applying Lemma 3.1 to $B^{(\varepsilon)}$ we find that they are uniformly bounded on $C(M)$. Applying Lemma 3.2 to (3.10) we find that $A_0^{-\varepsilon}$ are uniformly bounded on $C(M)$. Hence $A^{-\varepsilon}$ are uniformly bounded on $C(M)$.

To show that $A^{-\varepsilon}$ are uniformly bounded on $C^k(M)$ we just need to show that if D_k is a partial differential operator of order k on M then $D_kA^{-\varepsilon}$ are uniformly bounded from $C^k(M)$ to $C(M)$. But

$$D_kA^{-\varepsilon} \ = \ A^{-\varepsilon}D_k \ + \ [D_k, A^{-\varepsilon}].$$

Now D_k is bounded from $C^k(M)$ to C_M, and $A^{-\varepsilon}$ is uniformly bounded from $C(M)$ to $C(M)$ so we have dealt with the term $A^{-\varepsilon}D_k$. The commutators $[D_k, A^{-\varepsilon}]$ are uniformly bounded in $\Psi\mathrm{DO}^{k-1}(M)$, and so by Lemma 3.1 they are uniformly bounded from $C^k(M)$ to $C(M)$.

Appendix.

Let M be a closed, manifold with metric g. Denote the volume element by dV. Let A be an operator of type $\Delta^{n/2}$ with Robin mass m. In this appendix, we compute the difference between the regularization

$$\operatorname{trace} A^{-1} \ = \ \int_M m\,dV$$

and the zeta function regularization defined as follows. Let $\lambda_1 \le \lambda_2 \le \dots$ be the non-zero eigenvalues of A and set

$$Z(s) \;=\; \sum_j \lambda_j^{-s}.$$

By Weyl's law, $Z(s)$ converges for $\Re s > 1$. Now $Z(s)$ has an analytic continuation to a meromorphic function of $s \in \mathbb{C}$ with a simple pole at $s = 1$. We define $\operatorname{trace}_\zeta A^{-1}$ to be the finite part of $Z(s)$ at $s = 1$, that is

$$\operatorname{trace}_\zeta A^{-1} \;=\; Z(s)\Big|_{s=1}^{\text{f.p.}} := \left.\frac{d}{ds}\right|_{s=1} (s-1)Z(s).$$

To compare these two regularizations of the trace of A^{-1}, let ϕ_j be the eigenfunction of A with eigenvalue λ_j which is normalized in $L^2(M)$. The Schwartz kernel of Δ^{-s} is given by

$$K(A^{-s}, p, q) \;=\; \sum_j \lambda_j^{-s} \phi_j(p)\phi_j(q),$$

where the sum converges in the distributional sense. Away from the diagonal $p = q$, the function $K(A^{-s}, p, q)$ is entire in s and smooth in (s, p, q). At the diagonal, $K(A^{-s}, p, p)$ has a analytic continuation to a meromorphic function and

$$Z(s) \;=\; \int_M K(A^{-s}p, p)\, dV(p). \tag{A.1}$$

Hence

$$\operatorname{trace} A^{-1} \;-\; \operatorname{trace}_\zeta A^{-1} \;=\; \int_M \left(m(p) \;-\; K(A^{-s}, p, p)\Big|_{s=1}^{\text{f.p.}} \right) dV(p) \;=\; \int_M c(p)\, dV(p), \tag{A.2}$$

where $c(p)$ is the difference in two different ways of regularizing $K(A^{-1}, p, p)$:

$$c(p) \;=\; \lim_{q \to p} \left(K(A^{-1}, p, q) \;+\; \frac{2n}{\gamma_n} \log d(p, q) \right) \;-\; K(A^{-s}, p, p)\Big|_{s=1}^{\text{f.p.}}. \tag{A.3}$$

The operator A^{-s} is a pseudodifferential operator whose symbol expansion can be computed in terms of the symbol of A. In particular, working in normal coordinates at the point p, the principal symbol of A^{-s} is $|\xi|^{-ns}$. If one is familiar with the construction of powers of an elliptic operator, see for example [Se], it is not difficult to show that only the principal symbol will contribute to the anomaly (A.3). Indeed, if we define $K(s, x)$ to be the kernel of this symbol, that is

$$K(s, x) \;=\; \frac{1}{(2\pi)^n} \int_{|\xi|>1} e^{-ix\cdot\xi} |\xi|^{-ns}\, d\xi,$$

and if $X(q)$ denotes the normal coordinate of q centered at p, then the function

$$K(A^{-s}, p, q) \;-\; K(s, X(q))$$

is continuous in (s, p, q) for q in a neighborhood of p and $s \ge 1$. Hence from (A.3) we have

$$c(p) \;=\; \lim_{x \to 0} \left(K(1, x) \;+\; \frac{2n}{\gamma_n} \log |x| \right) \;-\; K(s, 0)\Big|_{s=1}^{\text{f.p.}}. \tag{A.4}$$

Now for $s < 1$ the function $|\xi|^{-ns}$ defines a homogeneous distribution, and its inverse Fourier transform can be computed by duality. As in (3.9),

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}|\xi|^{-ns}\,d\xi \;=\; C(s)|x|^{n(s-1)}, \qquad\qquad C(s) = (2\pi)^{-n/2}\,2^{n(\frac{1}{2}-s)}\frac{\Gamma(n(1-s)/2)}{\Gamma(ns/2)}.$$

Hence computing in polar coordinates,

$$\begin{aligned} K(s,x) \;&=\; C(s)|x|^{n(s-1)} \;-\; \frac{1}{(2\pi)^n}\int_{|\xi|<1} e^{ix\cdot\xi}|\xi|^{-ns}\,d\xi \\ &=\; C(s)|x|^{n(s-1)} \;-\; \frac{|S^{n-1}|}{(2\pi)^n}\int_{r<1} r^{-ns+n-1}\,dr \;+\; O(|x|^2) \\ &=\; C(s)|x|^{n(s-1)} \;-\; \frac{|S^{n-1}|}{n(2\pi)^n(1-s)} \;+\; O(|x|^2) \\ &=\; C(s)|x|^{n(s-1)} \;+\; \frac{2}{\gamma_n(s-1)} \;+\; O(|x|^2), \end{aligned}$$

for small $|x|$. We see that

$$K(s,0)\big|_{s=1}^{\text{f.p.}} \;=\; 0.$$

We have

$$c(p) \;=\; \lim_{x\to 0}\left(C(s)|x|^{n(s-1)} \;+\; \frac{2}{\gamma_n(s-1)} \;+\; \frac{2n}{\gamma_n}\log|x|\right) \;=\; C(s)\big|_{s=1}^{\text{f.p.}}.$$

From (A.2),

$$\text{trace}\,A^{-1} \;-\; \text{trace}_\zeta\,A^{-1} \;=\; C(s)\big|_{s=1}^{\text{f.p.}}\,V. \tag{A.5}$$

Explicitly we compute

$$C(s)\big|_{s=1}^{\text{f.p.}} \;=\; \frac{1}{(4\pi)^{n/2}\Gamma(n/2)}\left(2\log 2 \;+\; \Gamma'(1) \;+\; \frac{\Gamma'(n/2)}{\Gamma(n/2)}\right). \tag{A.6}$$

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